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# Replica-symmetry breaking: discrete and continuous schemes in the Sherrington–Kirkpatrick model

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## Abstract

We study hierarchies of replica-symmetry-breaking solutions of the Sherrington–Kirkpatrick model. Stationarity equations for order parameters of solutions with an arbitrary number of hierarchies are set and the limit to infinite number of hierarchical levels is discussed. In particular, we demonstrate how the continuous replica-symmetry breaking scheme of Parisi emerges and how the limit to infinite-many hierarchies leads to equations for the order-parameter function of the continuous solution. The general analysis is accompanied by an explicit asymptotic solution near the de Almeida–Thouless instability line in the nonzero magnetic field.

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## 1. Introduction

The solution of the Sherrington–Kirkpatrick (SK) model, standing for a mean-field theory of Ising spin glasses, is now almost complete. It took more than 30 years from the introduction of the model [1] before we understood its solution and in particular its physical meaning. The core of the mean-field solution was set rather early by Parisi in his replica-symmetry breaking (RSB) scheme [2]. Parisi used, however, the replica trick and a formal procedure of breaking a symmetry in the (unphysical) replica space when the limit to zero number of mathematical replicas is performed. Since then theorists have striven hard to understand the physical meaning of the Parisi solution and to find alternative ways of its derivation in order to prove its completeness.

The effort paid off. We have now reached solid understanding of the physics behind the RSB solution [3] and there is a mathematical proof of exactness of the RSB construction in the SK model [4, 5]. This is not the only output of the extensive investigation of mean-field spin-glass systems. The statistical methods developed by studying the infinite-range spin-glass models found broad application in interdisciplinary fields such as informatics, optimization

and computational complexity, econophysics and biophysics and other frustrated complex and open systems [6, 7].

Although our understanding of the construction of mean-field theories of spin glasses on regular lattices is in global attributes satisfactory, there still remain a few issues that deserve a more detailed and specific clarification. One of such questions is the eventual form of the replica-symmetry-breaking solution. Any derivation of a stable equilibrium spin-glass state uses the so-called discrete RSB scheme with finite-many hierarchical levels of the order parameters. Parisi assumed that in the SK model one needs infinite number of RSB hierarchies and derived an implicit formula for the free energy of the SK model with a continuous order-parameter function. The latter is now considered as the exact solution of the SK model. Unfortunately, the Parisi formula for the averaged free energy of the SK model is only implicit. Moreover, the mathematical proof of exactness of the RSB construction does not specify whether a discrete or the continuous RSB scheme produces the maximal free energy. There are models, Potts [8] or  $p$ -spin glass models [11] among others, for which one-step RSB solution seems to be stable in a finite region of temperatures. It is hence important to understand when one should use the discrete RSB scheme with a finite number of order parameters and when the continuous limit is appropriate.

The aim of this paper is to analyse properties of the discrete RSB scheme in the SK model leading to solutions with a finite number of hierarchical levels of the order parameters. The emphasis is laid on the way the discrete scheme goes over to the continuous Parisi solution in the limit of infinite-many RSB hierarchies. We show that the continuous limit is a process during which we reduce the number of degrees of freedom to a single continuous order-parameter function on a compact interval. We derive the continuous limit of the stationarity equations maximizing the free energy with large but finite numbers of RSB hierarchies and thereby we obtain a functional equation for the order-parameter function from the Parisi solution. We first derive the equations in the continuous limit generally and then we illustrate the process of building the continuous limit on the asymptotic expansion near the de Almeida–Thouless (AT) instability line of the SK model in the external magnetic field. We explicitly evaluate the leading asymptotic contribution to the Parisi order-parameter function. There exist expansions of the Parisi solution around the critical temperature  $T_c$  without [9] as well as with an applied magnetic field [10]. Here we expand the RSB solution for both discrete and continuous versions but not around the critical temperature (zero magnetic field) but around the replica-symmetric (RS) solution.

## 2. Discrete replica-symmetry breaking scheme

There is no direct way to the Parisi free energy surpassing the concept of discrete replicas in one or another way. We can derive the continuous RSB solution of the SK model in a number of ways, but we always have to start up with a finite number of replicas, mathematical or real. One cannot avoid analytic continuation of discrete parameters and quantities defined on a finite-many times replicated phase space to continuously distributed quantities of the Parisi solution. This can, however, be done only for a specific structure of the replicated phase space, namely a vertical tree containing only child replicas within parental ones. Communication between replicas of the same generation materializes exclusively via one or more antecedent generations of replicas (common ancestors). The phase space forms a hierarchical ultrametric structure. It is then natural to start the investigation of the RSB construction with the discrete scheme.

**Table 1.** Comparison of our real-replica and the replica-trick notations for the set of the RSB order parameters.

Real-replica notation		Replica-trick notation
Overlap susceptibility $\chi_l$		Block order parameter $q_l$
$\chi_l$	=	$q_{K+1-l} - q_0$
Multiplicity of replica generations $m_l$		Replica block size $m_l$
Decreasing sequence		Increasing sequence
$m_l$	$\longleftrightarrow$	$m_{K+1-l}$

2.1. Hierarchical free energy and order parameters

A hierarchical character of the phase space of the SK model is expressed in a hierarchy of order parameters standing for generations of the replicated spins. Each generation is characterized by a pair of numbers  $m_l$  and  $q_l$  from the replica trick or by  $m_l$  and  $\chi_l$  from the real-replica approach of [12]. In the replica trick, parameters  $m_l$  stand for block sizes with the same value of parameter  $q_l$ . In the real-replica approach, parameters  $m_l$  stand for multiplicity of replica generations and  $\chi_l$  for local overlap susceptibilities of two different spin replicas. We use here the latter notation with a decreasing sequence of overlap susceptibilities  $\chi_l$  as well as a decreasing sequence of parameters  $m_l$ , see table 1 for a transcript of the two notations. A phase space with  $K$  hierarchies (KRSB) is characterized in addition to the RS order parameter  $q$  also by  $K$  pairs  $\{m_l, \chi_l\}$  with  $l = 1, \dots, K$ . All these parameters are determined from stationarity equations for a hierarchical free energy with  $K$  hierarchies.

It is not the set of pairs  $\{m_l, \chi_l\}$  that explicitly appears in the hierarchical free energy. In fact, we can construct a free-energy functional for either  $\chi_l$ ,  $\Delta m_l = m_{l-1} - m_l$  or  $m_l$ ,  $\Delta \chi_l = \chi_l - \chi_{l+1}$  for  $l = 1, \dots, K$  with boundary conditions  $m_0 = 1$  and  $\chi_{K+1} = 0$ . Both set of parameters  $m_l$  and  $\chi_l$  form a decreasing sequence. The general formula for the hierarchical free energy with the former pairs was derived in [13] while the latter in [12] that we use also in this paper.

The free energy with  $K$  hierarchies is characterized by  $2K + 1$  order parameters. It is the RS order parameter  $q$  and  $K$  pairs  $\{m_l, \Delta \chi_l\}$ ,  $l = 1, \dots, K$ . The averaged free-energy density with these order parameters reads

$$f^K(q, \Delta \chi_1, \dots, \Delta \chi_K; m_1, \dots, m_K) = -\frac{1}{\beta} \ln 2 - \frac{\beta}{4} \left( 1 - q - \sum_{i=1}^K \Delta \chi_i \right)^2 + \frac{\beta}{4} \sum_{l=1}^K m_l \Delta \chi_l \left[ 2 \left( q + \sum_{i=l+1}^K \Delta \chi_i \right) + \Delta \chi_l \right] - \frac{1}{\beta} \int_{-\infty}^{\infty} \mathcal{D}\eta \ln \mathcal{Z}_K. \tag{1}$$

A hierarchical structure of this free energy is evident from the way its interacting part  $\ln \mathcal{Z}_K$  is constructed. It is the final state in a sequence of partition functions defined inductively

$$\mathcal{Z}_l = \left[ \int_{-\infty}^{\infty} \mathcal{D}\lambda_l \mathcal{Z}_{l-1}^{m_l} \right]^{1/m_l}, \tag{2}$$

with the initial condition  $\mathcal{Z}_0 = \cosh \left[ \beta (h + \eta \sqrt{q} + \sum_{l=1}^K \lambda_l \sqrt{\Delta \chi_l}) \right]$ . We abbreviated the Gaussian differential  $\mathcal{D}\lambda \equiv d\lambda \exp\{-\lambda^2/2\}/\sqrt{2\pi}$ .

Free energy (1) is a generalization of 1RSB and 2RSB solutions obtained by Parisi. It is a generating functional for all physical quantities of a  $K$ -level hierarchical solution. The physical values of the order parameters  $q, \Delta \chi_1, m_1, \dots, \Delta \chi_K, m_K$  are determined from

stationarity equations maximizing free energy  $f^K$ . To represent the corresponding stationarity equations we introduce a set of hierarchical density matrices in the space of fluctuating random fields  $\lambda_l$ . We define  $\rho_l(\eta, \lambda_K, \dots, \lambda_l) = Z_{l-1}^{m_l} / \langle Z_{l-1}^{m_l} \rangle_{\lambda_l}$  where  $\langle X(\lambda_l) \rangle_{\lambda_l} = \int_{-\infty}^{\infty} \mathcal{D}\lambda_l X(\lambda_l)$ . We further introduce short-hand notations  $t \equiv \tanh[\beta(h + \eta\sqrt{q} + \sum_{l=1}^K \lambda_l \sqrt{\Delta\chi_l})]$  and  $\langle t \rangle_l(\eta; \lambda_K, \dots, \lambda_{l+1}) = \langle \rho_l \dots (\rho_1 t)_{\lambda_1} \dots \rangle_{\lambda_l}$ .

It is now a straightforward task to derive equations for the order parameters from the saddle-point equations of functional  $f^K$ . We obtain

$$q = \langle t \rangle_K^2 \Big|_{\eta}, \tag{3}$$

$$\Delta\chi_l = \langle \langle t \rangle_{l-1}^2 \rangle_K \Big|_{\eta} - \langle \langle t \rangle_l^2 \rangle_K \Big|_{\eta}, \tag{4}$$

$$m_l \Delta\chi_l = \frac{4}{\beta^2} \frac{\langle \ln Z_{l-1} \rangle_K \Big|_{\eta} - \langle \ln Z_l \rangle_K \Big|_{\eta}}{2(q + \sum_{i=l+1}^K \Delta\chi_i) + \Delta\chi_l} \tag{5}$$

where index  $l = 1, \dots, K$ .

The discrete RSB scheme does not determine a single solution of the original spin model, but rather a set of solutions labelled by the number of hierarchies explicitly taken into account. Parameter  $K$  is hence a free index that is not determined from the free energy. Its physical value is fixed by thermodynamic stability. We take so many hierarchies of replicas into account till we reach a stable or marginally stable solution. Stable solutions with  $K$  hierarchies obey  $K + 1$  stability conditions. They are a generalization of the de Almeida–Thouless stability condition of the replica-symmetric ( $K = 0$ ) solution [14]. They reflect non-negativity of eigenvalues of the spin-glass susceptibility [15] and read

$$\Lambda_K(l) = 1 - \beta^2 \left\langle \left\langle \left\langle 1 - t^2 + \sum_{i=1}^l m_i \left( \langle t \rangle_{i-1}^2 - \langle t \rangle_i^2 \right) \right\rangle_l^2 \right\rangle_K \right\rangle_{\eta} \geq 0. \tag{6a}$$

and

$$\Lambda_K(0) = 1 - \beta^2 \langle \langle (1 - t^2)^2 \rangle_K \rangle_{\eta} \geq 0. \tag{6b}$$

These stability conditions are a non-replica version of the stability criteria set on the RSB solution within the replica trick [16, 17].

With the above equations we are equipped with all the necessary tools for identifying (marginally) stable solutions of the SK model. In particular, we can decide whether only a finite number of replica generations is sufficient to reach a stable or marginally stable solution or whether we must go to infinite-many replicas and the Parisi continuous limit. Parisi deduced from 1RSB and 2RSB solutions and confirmed by a truncated model [2] that indeed we need infinite number of replica hierarchies. We have recently confirmed this conclusion by solving exactly stationarity equations (3)–(5) in the asymptotic region near the critical point in zero magnetic field [18]. We now extend this asymptotic solution by involving the nonzero external magnetic field.

## 2.2. Asymptotic solution near the de Almeida–Thouless instability line

Proximity of the instability line naturally introduces a small parameter that we use in an expansion of equations (3)–(5) for the order parameters. If we denote  $t_0 = \tanh[\beta(h + \eta\sqrt{q_0})]$ , where  $q_0 = \langle t_0^2 \rangle_{\eta}$  is the order parameter in the replica-symmetric solution, we can define the small parameter to be

$$\alpha = \beta^2 \langle (1 - t_0^2)^2 \rangle_{\eta} - 1 > 0. \tag{7}$$

It measures the distance from the AT line as well as a deviation from the RS solution. Since the magnetic field is nonzero, only  $\chi_l$  for  $l = 1, \dots, K$  and  $\Delta m_l$  for  $l = 2, \dots, K$  are small. The parameters  $q$  and  $m_l$  are not small unlike the case of zero magnetic field [18]. We determine the dominant asymptotic behaviour of the corrections to the RS solution in the small parameter  $\alpha$  for an arbitrary number of hierarchical levels  $K$ .

To derive the leading asymptotic behaviour we must expand equations (3)–(5) to first two nontrivial orders of the small parameter  $\alpha$ . We first need to know the two leading asymptotic orders of the parameter  $q_{\text{EA}} = q + \sum_{l=1}^K \Delta\chi_l = q + \chi_1$  to be able to determine the leading asymptotic behaviour of  $\Delta\chi_l$  and  $\Delta m_l$  for  $l > 1$ . It means, if we want to go beyond the one-step RSB solution.

The asymptotic limit of the RSB solutions with finite numbers of hierarchies practically amounts to an expansion in powers of differences  $\Delta\chi_l$ . Each integral over the random field  $\lambda_l$  must be expanded at least up to  $\Delta\chi_l^3$  to determine the leading asymptotic behaviour of the order parameters. One can rather easily calculate the leading orders of  $\chi_1$  and  $m_l$ . These two parameters do not depend in the leading asymptotic order on the number of hierarchies used and are determined from 1RSB. To see the dependence of the order parameters on  $K$  means to generate separate equations for single  $\Delta\chi_l$  with  $l > 1$ . We must, however, lift up a degeneracy in the stationarity equations and expand them up to  $\Delta\chi_l^4$ . It is a rather tedious task and we accomplished it with the aid of the programme MATHEMATICA. The expansion proceeds in the same manner we presented in [18]. We hence do not repeat the detailed steps of the expansion but rather summarize the principal findings.

We must first expand the RS parameter  $q$  to the two lowest nontrivial orders in  $\alpha$ . The solution is then used to determine the lowest asymptotic order of  $\chi_1$  and  $m_l$ . We obtain  $m_l = m + O(\alpha)$  with

$$m = \frac{2\langle t_0^2(1-t_0^2)^2 \rangle_\eta}{\langle (1-t_0^2)^3 \rangle_\eta} \quad (8)$$

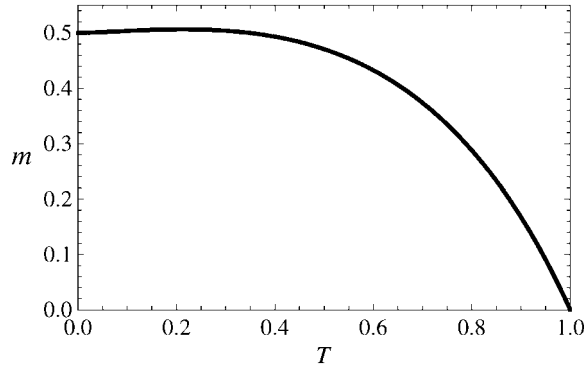
and

$$\chi_1 = \frac{\alpha}{2\beta^2 m} \frac{1}{1 - 3\beta^2 \langle t_0^2(1-t_0^2)^2 \rangle_\eta} + O(\alpha^2). \quad (9)$$

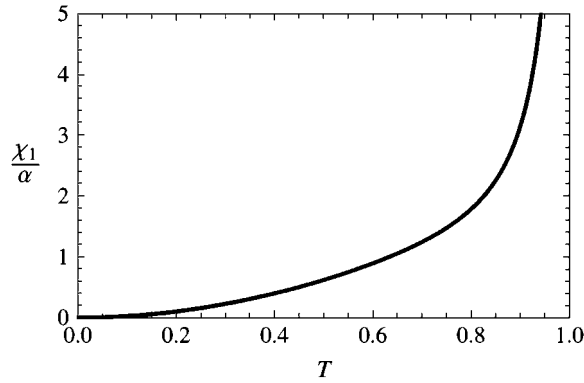
These two parameters do not depend on the number of hierarchical levels used. We remind that  $\chi_1 = q_{\text{EA}} - q$ . The parameter  $m$  is of order unity even at the boundary of the spin-glass phase (AT line) where the small parameter  $\alpha$  vanishes. Its temperature dependence at the AT line is plotted in figure 1.

The other parameter of the 1RSB solution,  $\chi_1$ , is proportional to the small parameter  $\alpha$  from equation (7) and vanishes at the boundary of the spin-glass phase. Its ratio  $\chi_1/\alpha$  at the AT line as a function of temperature is plotted in figure 2. The ratio diverges in zero magnetic field where  $m = 0$  and both  $\chi_1$  and  $m$  are linearly proportional to  $\theta = (T_c - T)/T_c$ , while  $\alpha \sim 2\theta^2$  [18].

It is interesting to note that the asymptotic solution near the AT line reduces to 1RSB in nonzero magnetic fields. The RS solution can be asymptotically correct in the leading order only in zero field where the parameter  $m_1 = 0$ . In nonzero fields one has to go to 1RSB even in the lowest asymptotic order below the instability line. The existence of the AT line hence indicates a replica-symmetry breaking. The instability in the magnetic field does not however specify whether the discrete or the continuous RSB scheme applies in the low-temperature phase. Note also that condition  $\chi_1 > 0$  does not necessarily indicate a deviation from the RS solution. If  $m_1 = 0$  then  $q_{\text{EA}} = q + \chi_1 = q_{\text{RS}}$ . Only if both parameters  $\chi_1$  and  $m_1$  are simultaneously positive the physics of the RS solution is changed to 1RSB.



**Figure 1.** The limiting value of parameter  $m$  from 1RSB at the AT instability line as a function of temperature. The temperature scale was chosen so that  $T_c(h = 0) = 1$ .



**Figure 2.** Proportionality of the order parameter  $\chi_1$  to the small parameter  $\alpha$  along the AT line. The ratio diverges at the critical temperature  $T_c = 1$  as  $(T_c - T)^{-1}$  and vanishes at zero temperature as  $T^2$ .

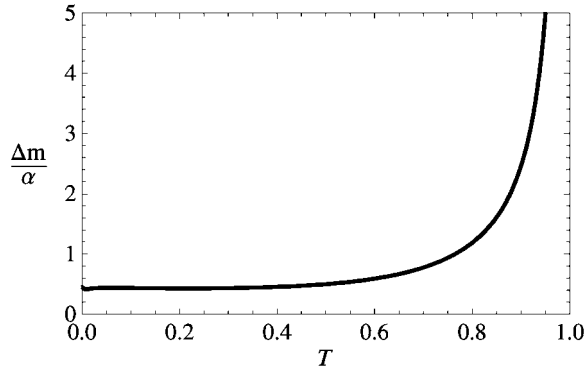
To disclose the leading asymptotic behaviour of each separate parameter  $\Delta\chi_l$  and  $\Delta m_l$  for  $l = 1, \dots, K$  we must go beyond 1RSB and the leading orders in parameters  $m$  and  $\chi_1$ . It is firstly the fourth order in  $\alpha$  in equation (5) from which we find that  $\Delta\chi_l \doteq \chi_1^1/K$  and

$$m_l^K \doteq m_1^1 + \frac{K + 1 - 2l}{K} \Delta m \tag{10a}$$

where we added a superscript to specify the number of hierarchical levels used to determine the order parameters  $\chi_l, m_l$ . Further on, we introduced a parameter independent of the number of hierarchies  $\Delta m = m_1^2 - m_2^2$ . This parameter has an explicit asymptotic representation:

$$\Delta m \doteq \frac{\beta^2 \chi_1 \langle (1 - t_0^2)^2 (2(1 - 3t_0^2)^2 + 3(t_0^2 - 1)m(8t_0^2 + (t_0^2 - 1)m)) \rangle_\eta}{\langle (1 - t_0^2)^3 \rangle_\eta} \tag{10b}$$

Both parameters  $\chi_1$  and  $\Delta m$  are linearly proportional to  $\alpha$ . The former, however, exists already in 1RSB, while the latter first emerges in 2RSB. Since they do not depend on the



**Figure 3.** The leading-order difference  $\Delta m$  factorized by the small parameter  $\alpha$  to make it of order unity along the AT line. It diverges at the critical temperature  $T_c = 1$  as  $(T_c - T)^{-1}$ .

number of hierarchies used and the latter determines a uniform distribution of parameters  $m_l$  for  $l = 3, \dots, K$ , we demonstrated that all characteristic features of the asymptotic solution near the AT instability line are contained already in 2RSB. What was, however, highly nontrivial was to unveil equidistant distributions of both parameters  $\chi_l$  and  $m_l$ . Temperature dependence of the ratio  $\Delta m/\alpha$  along the AT line is plotted in figure 3.

Finally we evaluated the instability conditions from equations (6a) and (6b). They all coincide in the leading asymptotic order in  $\alpha$ . We find that all the discrete RSB solutions are unstable. Instability of the discrete scheme is measured by the small parameter  $\alpha$ . The RS ( $K = 0$ ) solution has the instability expressed via the AT condition

$$\Lambda_0 = 1 - \beta^2 \langle (1 - t_0^2) \rangle_\eta = -\alpha. \tag{11a}$$

The RSB solutions ( $K \geq 1$ ) improve upon stability of the RS solution in that their instability is proportional to  $\alpha^2$ . We derived the following explicit expression

$$\Lambda_K = -\frac{2\beta^2}{3K^2} \frac{\chi_1 \Delta m}{m + 2} \tag{11b}$$

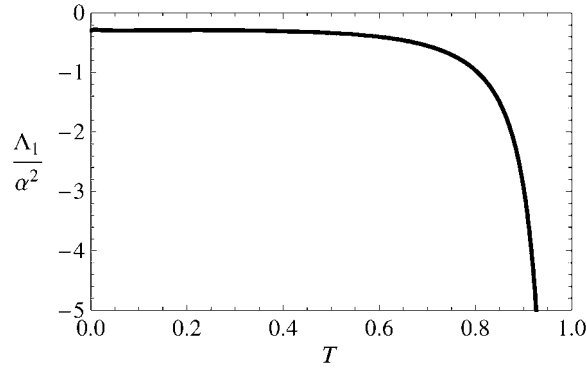
where  $\Delta m$  was defined in equation (10b). The difference in the order of magnitude in the instability of the RS and RSB solutions is caused by the existence of the nonzero parameter  $m$  at the AT line where  $\chi_l = 0$ . It is hence impossible for the RS solution without  $m$  to reproduce the exact solution in the nonzero magnetic field even asymptotically with  $\alpha \rightarrow 0$ .

The stability conditions of the discrete RSB solutions manifest that only the continuous limit with  $K \rightarrow \infty$  becomes marginally stable. The instability of 1RSB is plotted in figure 4. The leading-order term diverges at the critical temperature  $T_c = 1$ , since the instabilities of the RS as well as of the discrete RSB solutions in zero magnetic field are proportional to  $\alpha \propto (T_c - T)^2/T_c^2$  [18].

### 3. Continuous replica-symmetry breaking

The above asymptotic solution corroborates the conclusion of earlier calculations in zero magnetic field that to reach a stable and consistent solution we need infinite-many hierarchical levels in free energy (1). It is not generally guaranteed that the limit to infinite hierarchies of the discrete RSB scheme must lead to the continuous solution. The distribution of parameters  $\Delta\chi_l = \chi_l/K$  calculated in the preceding section explicitly manifests that the Parisi solution





**Figure 4.** Leading asymptotic contribution to the instability of 1RSB at the AT line. It diverges at the critical temperature as  $(T_c - T)^{-2}$ .

is the only possible marginally stable solution of the SK model in the external magnetic field near the AT line. Moreover, one of us has recently shown that the Parisi solution with a continuous order-parameter function can always be constructed and is marginally stable in the entire spin-glass phase [19]. Here we show that the saddle-point equations of the free energy from [19] result also from the continuous limit of stationarity equations (3)–(5) of the discrete scheme.

### 3.1. Homogeneous limit to infinite number of hierarchies

The basic assumption of the continuous limit of the discrete RSB scheme with the number of hierarchies  $K \rightarrow \infty$  is a uniform distribution of the differences  $\Delta\chi_l$ . That is, they are independent of  $l$  and are proportional to  $K^{-1}$ . We then can introduce a differential  $dx = \lim_{K \rightarrow \infty} \chi_l / K$ . It is not, however, necessary that all parameters  $\Delta\chi_l$  are equal in the asymptotic limit  $K \rightarrow \infty$  to end up in a continuous theory. The differences may vary by factors of order unity.

It is convenient to map the hierarchy indices  $l$  on interval  $[0, 1]$  by introducing a continuous variable  $x = \lim_{K \rightarrow \infty} (K - l) / K$ . The continuous version of the hierarchical free energy results from a process in which we systematically neglect all higher than linear orders of the differential  $dx$  [20]. It means that in the continuous limit we take into account only the second moments of the Gaussian integrations over the auxiliary fields  $\lambda_l$ . We first apply this procedure to the interacting free energy (1) and then to stationarity equations (3)–(5).

We denote  $g_l \equiv \ln \mathcal{Z}_l$ . Using the notation from the preceding section we obtain by cutting the expansion of equation (2) at the order  $O(\Delta\chi_l)$

$$\begin{aligned}
 g_l &= \ln \langle \mathcal{Z}_{l-1}^{m_{l-1}} \rangle_{\lambda_l}^{1/m_{l-1}} \rightarrow \frac{1}{m_{l-1}} \ln \left\{ \mathcal{Z}_{l-1}^{m_{l-1}} \left[ 1 + \frac{m_{l-1}}{2} \Delta\chi_l (g_{l-1}'' + m_{l-1} g_{l-1}^{\prime 2}) \right] \right\} \\
 &= g_{l-1} + \frac{\Delta\chi_l}{2} (g_{l-1}'' + m_{l-1} g_{l-1}^{\prime 2}).
 \end{aligned} \tag{12}$$

We denoted  $g_l' \equiv \partial g_l / \partial h$ . The derivatives with respect to the magnetic field stand for the action of the fluctuating field  $\lambda_l$  when only the second moment contributes to the integral. In

the continuous limit we obtain the Parisi differential equation by replacing the hierarchy index  $l$  by the continuous variable  $x$

$$\frac{\partial g(x, h)}{\partial x} = \frac{\dot{\chi}(x)}{2} \left[ \frac{\partial^2 g(x, h)}{\partial h^2} + m(x) \left( \frac{\partial g(x, h)}{\partial h} \right)^2 \right]. \quad (13)$$

We denoted  $\dot{\chi}(x) \equiv d\chi(x)/dx$ . The right-hand side of equation (13) has the opposite sign to the original Parisi equation, which is caused by a different assignment of the continuous variable  $x$  to the hierarchy index. Parisi used  $x^P = \lim_{K \rightarrow \infty} l/K$ .

To derive the continuous limit of the stationarity equations we must first find a reduction of the density matrix  $\rho_l$  in the limit of infinite hierarchical levels. Since only a linear term in  $\Delta\chi_l$  contributes and the density matrix is normalized to unity we have

$$\rho_l \rightarrow 1 + m_{l-1} g'_{l-1} \Delta\chi_l \frac{\partial}{\partial h}. \quad (14)$$

The operator of the derivative with respect to the magnetic field stands for the random variable  $\lambda_l$ . Integrals over this random variable of functions  $f(\lambda_l, h)$  weighted with the density matrix  $\rho_l$  reduce in the continuous limit to

$$\langle \rho_l f(\lambda_l, h) \rangle_{\lambda_l} = f(0, h) + m_{l-1} \Delta\chi_l g'_{l-1} \frac{\partial f(0, h)}{\partial h} + \frac{\Delta\chi_l}{2} \frac{\partial^2 f(0, h)}{\partial h^2}. \quad (15)$$

We need to evaluate multiple integrals over a number of random fields  $\lambda_l$ . We denote

$$f_{l,i} = \langle \rho_{l+i} \dots \langle \rho_{l+1} f \rangle_{\lambda_{l+1}} \dots \rangle_{\lambda_{l+i}} \quad (16a)$$

and using rule (15) we end up with

$$f_{l,i+1} = \langle \rho_{l+i+1} f_{l,i} \rangle_{\lambda_{l+i+1}} = f_{l,i} + \Delta\chi_{l+i+1} \left[ m_{l+i} g'_{l+i} f'_{l,i} + \frac{1}{2} f''_{l,i} \right]. \quad (16b)$$

The increment in the second index can again be represented in the continuous limit via a differential equation

$$\frac{\partial f_l(X, h)}{\partial X} = \dot{\chi}(X) \left[ m_l(X) \frac{\partial g(X, h)}{\partial h} \frac{\partial f_l(X, h)}{\partial h} + \frac{1}{2} \frac{\partial^2 f_l(X, h)}{\partial h^2} \right]. \quad (17)$$

The solution of the above equation can be represented in form of a ‘time-ordered’ exponential with differential operators [19]:

$$f_x(X, h) = T_y \exp \left\{ \int_x^X dy \dot{\chi}(y) \left[ \frac{1}{2} \partial_{\bar{h}}^2 + m(y) g'(y, h + \bar{h}) \partial_{\bar{h}} \right] \right\} f_x(0, h + \bar{h}) \Big|_{\bar{h}=0}. \quad (18)$$

The ordering operator  $T_y$

$$T_y \exp \left\{ \int_a^b dy \widehat{O}(y) \right\} \equiv 1 + \sum_{n=1}^{\infty} \int_a^b dy_1 \int_a^{y_1} dy_2 \dots \int_0^{y_{n-1}} dy_n \widehat{O}(y_1) \dots \widehat{O}(y_n) \quad (19)$$

orders products of  $y$ -dependent non-commuting operators from left to right in a  $y$ -decreasing succession. It is a standard tool used in many-body quantum theory to represent the time-dependent perturbation expansion. It is easy to check that function  $f_x(X, h)$  obeys equation (17).

With the aid of solution (18) we can represent any physical quantity in the continuous limit. First among them are the equations for the order parameters.

### 3.2. Stationarity equations and stability conditions

To derive an equation for the RS order parameter  $q$  we simply put  $f_x(0, h) = t(h) \equiv \tanh(\beta h)$ . We obtain from equation (18)

$$t(X, h) = T_y \exp \left\{ \int_0^X dy \dot{\chi}(y) \left[ \frac{1}{2} \partial_{\bar{h}}^2 + m(y) g'(y, h + \bar{h}) \partial_{\bar{h}} \right] \right\} t(h + \bar{h}) \Big|_{\bar{h}=0}. \quad (20)$$

Using this representation in equation (3) we reach at a generalization of the RS relation

$$q = \langle t(1, h + \eta \sqrt{q})^2 \rangle_{\eta}. \quad (21)$$

To evaluate the right-hand side of equation (4) we realize that

$$\langle \rho_l \langle t \rangle_{l-1}^2 \rangle_{\lambda_l} - \langle \rho_l \langle t \rangle_{l-1} \rangle_{\lambda_l}^2 \rightarrow \Delta \chi_l \langle t \rangle_{l-1}^2$$

from which we find with the aid of integral representation (18)

$$\begin{aligned} \dot{\chi}(x) = \dot{\chi}(x) \left\langle T_y \exp \left\{ \int_x^1 dy \dot{\chi}(y) \left[ \frac{1}{2} \partial_{\bar{h}}^2 + m(y) g'(y, h + \bar{h}) \partial_{\bar{h}} \right] \right\} \right. \\ \left. \times (\partial_{\bar{h}} t(x, h_{\eta} + \bar{h}))^2 \Big|_{\bar{h}=0} \right\rangle_{\eta}. \end{aligned} \quad (22)$$

We denoted  $h_{\eta} = h + \eta \sqrt{q}$ . Equation (22) is fulfilled for all  $x \in [0, 1]$ . It essentially determines the functional dependence  $\dot{\chi}(x)$ . We know from the discrete scheme that  $\dot{\chi}(x) > 0$ .

The last relation to be rewritten in the continuous limit is equation (5). It is not difficult to reach a representation

$$\begin{aligned} m(x)(q + \chi(1) - \chi(x)) = m(x) \left\langle T_y \exp \left\{ \int_x^1 dy \dot{\chi}(y) \right. \right. \\ \left. \left. \times \left[ \frac{1}{2} \partial_{\bar{h}}^2 + m(y) g'(y, h + \bar{h}) \partial_{\bar{h}} \right] \right\} t(x, h_{\eta} + \bar{h})^2 \Big|_{\bar{h}=0} \right\rangle_{\eta} \end{aligned} \quad (23)$$

that now determines the functional dependence  $m(x)$ . Again from the discrete RSB scheme we know that  $m(x) < 0$ . The two functional equations (22) and (23) allow for a trivial solution making thereby the RS solution part of the general RSB scheme.

The derived equations for the continuous version of the order parameters from the discrete RSB scheme enable us to understand how we get rid of one functional order parameter. Namely, the function  $\dot{\chi}(x)$  appears in all physical quantities only under integrals over the index variable  $x$ . We hence can redefine the differential  $dx \rightarrow d\chi = dx \dot{\chi}(x)$ , since  $\dot{\chi}(x) > 0$ . We do not need to know the point-wise dependence  $\dot{\chi}(x)$  to determine physical properties of the low-temperature spin-glass state. We hence can transform the defining interval  $x \in [0, 1]$  to a new one  $\chi \in [0, X]$ , where  $X = \chi(1) \leq 1$ . The largest value of  $\chi(x)$  is the only parameter we need to know from this function. It must be determined from a stationarity equation and is related to the Edwards–Anderson parameter by an equation  $q_{EA} = q + X$ . When we resign on the explicit dependence  $\dot{\chi}(x)$  we also have to disregard equation (22). Only stationarity equations for  $q$  and  $m(\chi)$ , equations (21) and (23), respectively, remain then relevant. They coincide with the equations derived from stationarity conditions imposed upon the Parisi free energy in [19].

Last but not least we have to find the continuous version of stability conditions (6a) and (6b). It is again straightforward to use the above results and integral representation (18) to

arrive at

$$\Lambda(x) = 1 - \beta^2 \left\langle T_y \exp \left\{ \int_x^1 dy \dot{\chi}(y) \left[ \frac{1}{2} \partial_{\bar{h}}^2 + m(y) g'(y, h + \bar{h}) \partial_{\bar{h}} \right] \right\} \left[ 1 - t(x, h_\eta + \bar{h})^2 + \int_0^x dz \dot{\chi}(z) m(z) T_y \exp \left\{ \int_z^x dy \dot{\chi}(y) \left[ \frac{1}{2} \partial_{\bar{h}}^2 + m(y) g'(y, h + \bar{h}) \partial_{\bar{h}} \right] \right\} \times (\partial_{\bar{h}} t(z, h_\eta + \bar{h}))^2 \right]_{\bar{h}=0} \right\rangle_\eta \geq 0. \quad (24)$$

It was shown in [19] that equality in equation (24) can be derived from a total derivative of equation (23) with respect to variable  $x$ . It means that if equation (23) is fulfilled for all variables  $x \in [0, 1]$ , then  $\Lambda(x) = 0$  as well. We thereby confirmed that the Parisi continuous RSB solution is marginally stable. The spin-glass susceptibility has zero eigenvalue but no negative ones.

### 3.3. Asymptotic solution near the de Almeida–Thouless instability line

We now explicitly solve the equations for the order parameters of the continuous RSB scheme in the asymptotic limit to the AT line. We know from the discrete version that  $\dot{\chi}(x) = 1$  for  $x \in [0, X]$  and vanishes elsewhere. We introduce a new dimensionless variable  $\lambda = x/X$  that spans interval  $[0, 1]$ . The physical parameter  $X$  serves as an expansion parameter in the asymptotic region near the AT instability line.

In the continuous limit we have two basic equations to solve. It is equation (21) for the RS order parameter  $q$  and equation (23) from which we determine  $X$  and  $m(\lambda)$ . It can be shown that for  $x \leq X$  equation (22) is a total derivative of equation (23). There is thus no inconsistency if we disregard equation (22) as a stationarity equation for the free energy of the continuous RSB solution. We know from the preceding subsection that the total derivative of equation (23) with respect to  $x$  expresses a marginal stability of the continuous RSB state.

The asymptotic solution near the AT line is a polynomial in variable  $X$ . We hence expand all quantities to a necessary order in this parameter. The  $T$ -exponential on the right-hand side of equation (23) must be expanded to  $X^3$ . Further on, the order-parameter function becomes also a polynomial in  $X$ . The relevant order parameters are then expanded as follows:

$$q = q_0 + Xq'_1 + X^2q'_2, \quad (25a)$$

$$m(\lambda) = m_0 + X\lambda m'_1. \quad (25b)$$

We first use equation (21) to simplify equation (23) and then expand its right-hand side to  $X^3$ . Simultaneously we make use of the expansion of the order parameters from equations (25a) and (25b). We obtain a cubic polynomial in  $\lambda$ . Coefficients at each power of  $\lambda$  must vanish and we have three equations for parameters  $m_0$ ,  $m'_1$  and  $X$ . Parameters  $q'_1$  and  $q'_2$  are determined from an expansion of equation (21). The explicit solution was calculated with the aid of the programme MATHEMATICA and reads

$$q_0 = \langle t_0^2 \rangle_\eta, \quad (26a)$$

$$q'_1 = - \frac{2\beta^2(1 - m_0) \langle t_0^2(1 - t_0^2) \rangle_\eta}{1 - \beta^2 \langle (1 - t_0^2)(1 - 3t_0^2) \rangle_\eta}, \quad (26b)$$

$$q'_2 = - \frac{\langle Q_2 \rangle_\eta}{1 - \beta^2 \langle (1 - t_0^2)(1 - 3t_0^2) \rangle_\eta}. \quad (26c)$$

Here we denoted

$$Q_2 = (1 - t_0^2) [(\beta^2(m_0 - 1)(-7t_0^2 + (5t_0^2 - 3)m_0 + 5) - m'_1)t_0^2 + (15(t_0^2 - 1)t_0^2 + 2)\beta^2 q_1^2 - 2(10t_0^4 - 9t_0^2 + 1)\beta^2(m_0 - 1)q_1'].$$

The above solution is then used in the equations for  $X$ ,  $m_0$ ,  $m'_1$ . These three equations read

$$1 = \left( (1 - t_0^2)^2 \beta^2 (1 - 2X\beta^2(m_0(2t_0^2 - (37t_0^4 + 22t_0^2 - 1)X\beta^2) - Xm'_1 t_0^2 + (7t_0^2 - 3)X\beta^2 m_0^2 t_0^2) + X\beta^2 (10t_0^2 + (105t_0^4 - 80t_0^2 + 7)X\beta^2 - 2)) \right)_\eta, \quad (27a)$$

$$0 = \left( (1 - t_0^2)(-2t_0^2 - 2(21t_0^4 - 14t_0^2 + 1)X\beta^2 + m_0(6(t_0^2 - 1)X\beta^2 m_0 t_0^2 + (-9t_0^4 + 20t_0^2 - 3)X\beta^2 + 1 - t_0^2)) \right)_\eta, \quad (27b)$$

$$0 = \left( (1 - t_0^2)(m'_1(1 - t_0^2) + 2\beta^2(1 - 3t_0^2)^2 + 3(t_0^2 - 1)\beta^2 m_0(8t_0^2 + (t_0^2 - 1)m_0)) \right)_\eta. \quad (27c)$$

We determine  $m'_1$  from equation (27c),  $m_0$  then from equation (27b). The two parameters we finally use in equation (27a) from which we calculate  $X$ . To find the leading asymptotic behaviour of  $X$  we have to expand it in powers of the initial small parameter  $\alpha$  measuring the depth of penetration into the spin-glass phase. Parameter  $\alpha$  was defined in equation (7) and emerges in equation (27a) as an absolute,  $X$ -independent term. The solutions for  $X$  and  $m_0$  from the continuous RSB scheme then coincide in the leading order in  $\alpha$  with the result for  $\chi_1$  and  $m$  from the discrete 1RSB solution, equations (9) and (8), respectively. The last parameter  $m'_1$  has an explicit representation

$$m'_1 = -\frac{2\Delta m}{X} \quad (28)$$

where again parameter  $\Delta m$  was already determined within the discrete 2RSB scheme in equation (10b). The asymptotic limit of the full solution near the AT line is hence completely determined by the parameters from the two-step RSB solution.

#### 4. Conclusions

We studied in this paper the behaviour of the replica-symmetry breaking solutions in the discrete and continuous schemes. We started with the discrete one with  $K$  hierarchies and  $2K + 1$  order parameters  $q$ ,  $\Delta\chi_1$ ,  $m_1$ ,  $\dots$ ,  $\Delta\chi_K$ ,  $m_K$ . Equations for these order parameters were derived from a local maximum of a free energy and were explicitly solved in the asymptotic limit to the AT instability line. This calculation served as an explicit manifestation of the way the Parisi continuous RSB solution in the nonzero magnetic field is approached in the limit  $K \rightarrow \infty$ . We found that unlike in zero magnetic field, the RS solution is never, even asymptotically, stable in the nonzero field below the AT line. The full solution in the low-temperature spin-glass phase reduces near the AT line to the one-step RSB solution. We found that  $\sum_{l=1}^K \Delta\chi_l = \chi_1$  and  $m_l$  do not depend in the leading asymptotic order on the number of hierarchies  $K$  and are exactly determined by 1RSB. Further on, we demonstrated that neither  $\Delta m = K(m_{l-1} - m_l)/2$  for  $l \geq 2$  depends on the number of hierarchies  $K$  used. The characteristic parameters of the full asymptotic solution are completely set by 2RSB. There is no other parameter characterizing the asymptotic limit to the AT line. Increasing the number of hierarchies in free energy (1) does not change the values  $\chi_1 = \sum_{l=1}^K \Delta\chi_l$  and  $\Delta m$ . The new added order parameters  $\chi_l$ ,  $m_l$  for  $l = 3, \dots, K$  are equidistantly distributed between the edge values  $0 \leq \chi_l < \chi_1$  and  $m_1 - \Delta m < m_l < m_1 + \Delta m$ , where  $\chi_1$  and  $m_1$  are

calculated in 1RSB. These results explicitly prove that the discrete RSB scheme goes over in the limit of infinite number of hierarchies to the Parisi continuous RSB solution.

We analysed the behaviour of the discrete RSB scheme in the limit  $K \rightarrow \infty$  also generally. We performed this limit explicitly in the stationarity equations maximizing the free energy with finite-many hierarchical levels. We derived in this way a set of equations for the order parameters in the continuous limit. The equations for the order parameters from the discrete RSB scheme go over in the continuous limit to two functional equations for order-parameter functions  $\check{\chi}(x)$  and  $m(x)$ . Since the former function comes up only under integrals over the index variable  $x \in [0, 1]$ , its point-wise behaviour is irrelevant for the physical quantities. The only significant information from  $\check{\chi}(x)$  is an integral  $\int_0^1 dx \check{\chi}(x) = X$ . We hence can disregard the defining equations for  $\check{\chi}(x)$  and take explicitly into account only equations for the RS parameter  $q$  and for function  $m(x)$ . Parameter  $X$  is determined from a combination of the two equations. We do not lose any information by neglecting the defining equation for  $\check{\chi}(x)$  that was shown to be a total derivative of the equation for  $m(x)$ . This feature expresses a degeneracy of the stationarity equations in the discrete RSB scheme. The equations resulting from the continuous limit of the equations from the discrete scheme are identical with those derived directly from the Parisi free energy via a saddle point in [19]. It means that the local maximum of the Parisi free energy is a limit of local maxima of discrete hierarchical free energies (1) when  $K \rightarrow \infty$ . The continuous limit is analytical and all physical quantities can be defined and calculated either directly from the Parisi free-energy functional of [19] or from the limit  $K \rightarrow \infty$  of quantities introduced in the discrete scheme with free energy (1).

The explicit asymptotic solution of the discrete KRSB scheme enabled the calculation of its thermodynamic stability. We found that in the leading asymptotic order of the discrete scheme all the stability conditions (6a) and (6b) coincide. Their value is negative for any finite number of hierarchies  $K$  and approaches zero as  $K^{-2}$ . The continuous scheme is then marginally stable with no negative eigenvalue of the spin-glass susceptibility. The RS solution is asymptotically stable in the leading order below the critical temperature only in zero magnetic field. In the nonzero magnetic field the spin-glass state goes over asymptotically to the one-step replica-symmetry-breaking solution ( $K = 1$ ) that is marginally stable in the leading order near the AT instability line.

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## References

- [1] Sherrington D and Kirkpatrick S 1975 *Phys. Rev. Lett.* **35** 1972
- [2] Parisi G 1980 *J. Phys. A: Math. Gen.* **13** L115, 1101, 1887
- [3] Parisi G 2007 *Preprint 0706.0094v1* (also part of [7])
- [4] Guerra F 2003 *Commun. Math. Phys.* **233** 1
- [5] Talagrand M 2006 *Ann. Math.* **163** 221
- [6] Nishimori H 2001 *Statistical Physics of Spin Glasses and Information Processing* (Oxford: Oxford University Press)
- [7] Bouchaud J P, Mézard M and Dalibard J (ed) 2007 *Complex Systems (Lecture Notes of the Les Houches Summer School 2006)* vol LXXXV (Amsterdam: Elsevier)
- [8] Gross D J, Kanter I and Sompolinsky H 1985 *Phys. Rev. Lett.* **55** 304
- [9] Crisanti A and Rizzo T 2002 *Phys. Rev. E* **65** 046137

- [10] Crisanti A, Rizzo T and Temesvari T 2003 *Eur. J. Phys. B* **33** 203
- [11] Gardner E 1985 *Nucl. Phys. B* **240** 747
- [12] Janiš V 2005 *Phys. Rev. B* **71** 214403
- [13] Dotsenko V 2001 *Introduction to the Replica Theory of Disordered Statistical Systems* (Cambridge: Cambridge University Press)
- [14] de Almeida J R L and Thouless D J 1978 *J. Phys. A: Math. Gen.* **11** 983
- [15] Janiš V 2006 *Phys. Rev. B* **74** 054207
- [16] de Dominicis C and Kondor I 1983 *Phys. Rev. B* **27** 606
- [17] Goltsev A V 1983 *J. Phys. A: Math. Gen.* **16** 1337
- [18] Janiš V and Klíč A 2006 *Phys. Rev. B* **74** 054410
- [19] Janiš V 2007 *Preprint* [0711.1648v1](#)
- [20] Duplantier B 1981 *J. Phys. A: Math. Gen.* **14** 283